

# Midterm COMP 2804

October 23, 2015

- All questions must be answered on the scantron sheet.
- Write your name and student number on the scantron sheet.
- You do not have to hand in this examination paper.
- Calculators are allowed.

**Marking scheme:** Each of the 17 questions is worth 1 mark.

- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$
- Newton:  $(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^{n-k} y^k$ .

1. The Carleton Computer Science Society has a Board of Directors consisting of a President, two Vice-Presidents, and a five-person Advisory Board. The President cannot be Vice-President and cannot be on the Advisory Board. A Vice-President cannot be on the Advisory Board. Let  $n$  be the number of students in Carleton's Computer Science program, where  $n \geq 8$ . In how many ways can a Board of Directors be chosen?
  - (a)  $n \binom{n}{2} \binom{n}{5}$
  - (b)  $(n-2) \binom{n}{2} \binom{n-2}{5}$
  - (c)  $(n-5) \binom{n}{2} \binom{n-1}{5}$
  - (d)  $(n-7) \binom{n}{2} \binom{n-2}{5}$
2. Let  $S$  be a set of 25 elements and let  $x$ ,  $y$ , and  $z$  be three distinct elements of  $S$ . What is the number of subsets of  $S$  that contain both  $x$  and  $y$ , but do not contain  $z$ ?
  - (a)  $2^{25} - 2^{22}$
  - (b)  $2^{25} - 2^{24} + 2^{23}$
  - (c)  $2^{22}$
  - (d)  $2^{23}$
3. Let  $A$  be a set of 6 elements and let  $B$  be a set of 13 elements. How many one-to-one (i.e., injective) functions  $f : A \rightarrow B$  are there?
  - (a)  $5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13$
  - (b)  $6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13$
  - (c)  $7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13$
  - (d)  $8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13$
4. For any integer  $n \geq 2$ , let  $S_n$  be the number of bitstrings of length  $n$  in which the first bit is not equal to the last bit. Which of the following is true?
  - (a)  $S_n = 2^{n-2}$
  - (b)  $S_n = 2^{n-1}$
  - (c)  $S_n = 2^n - 2^{n-2}$
  - (d)  $S_n = 2^n - 2^{n-1} + 2^{n-2}$

5. Consider strings of length 99 consisting of the characters  $a$ ,  $b$ , and  $c$ . How many such strings are there that start with  $abc$  or end with  $bbb$ ?

- (a)  $3^{96} + 3^{96}$
- (b)  $3^{99} - 2 \cdot 3^{96}$
- (c)  $2 \cdot 3^{96} - 3^{93}$
- (d) None of the above.

6. What does

$$\sum_{k=1}^m \binom{m}{k}$$

count?

- (a) The number of non-empty subsets of a set of size  $m$ .
  - (b) The number of subsets of a set of size  $m$ .
  - (c) The number of bitstrings of length  $m$  having exactly  $k$  many 1s.
  - (d) None of the above.
7. In the city of `SHORTLASTNAME`, every person has a last name consisting of two characters, the first one being an uppercase letter and the second one being a lowercase letter. What is the minimum number of people needed so that we can guarantee that at least 4 of them have the same last name?

- (a)  $3 \cdot 26^2$
- (b)  $4 \cdot 26^2$
- (c)  $3 \cdot 26^2 + 1$
- (d)  $4 \cdot 26^2 + 1$

8. What is the coefficient of  $x^{81}y^7$  in the expansion of  $(3x - 17y)^{88}$ ?

- (a)  $\binom{88}{7} \cdot 3^{81} \cdot 17^7$
- (b)  $-\binom{88}{7} \cdot 3^{81} \cdot 17^7$
- (c)  $\binom{88}{7} \cdot 3^7 \cdot 17^{81}$
- (d)  $-\binom{88}{7} \cdot 3^7 \cdot 17^{81}$

9. How many solutions are there to the equation  $x_1 + x_2 + x_3 + x_4 = 55$ , where  $x_1 \geq 0$ ,  $x_2 \geq 0$ ,  $x_3 \geq 0$ , and  $x_4 \geq 0$  are integers?

- (a)  $\binom{58}{3}$
- (b)  $\binom{58}{4}$
- (c)  $\binom{59}{3}$
- (d)  $\binom{59}{4}$

10. The function  $f : \mathbb{N} \rightarrow \mathbb{N}$  is defined by

$$\begin{aligned} f(0) &= 7 \\ f(n) &= f(n-1) + 10n - 6 \text{ for } n \geq 1 \end{aligned}$$

What is  $f(n)$ ?

- (a)  $f(n) = 4n^2 - 2n + 7$
  - (b)  $f(n) = 4n^2 - n + 7$
  - (c)  $f(n) = 5n^2 - 2n + 7$
  - (d)  $f(n) = 5n^2 - n + 7$
11. Let  $S_n$  be the number of bitstrings of length  $n$  that contain the substring 0000. Which of the following is true?

- (a)  $S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4}$
- (b)  $S_n = S_{n-1} + S_{n-2} + S_{n-3} + S_{n-4} + 2^{n-4}$
- (c)  $S_n = S_{n-1} + S_{n-2} + S_{n-3}$
- (d)  $S_n = S_{n-1} + S_{n-2} + S_{n-3} + 2^{n-3}$

12. Let  $n \geq 1$  be an integer and let  $S_n$  be the number of ways in which  $n$  can be written as a sum of 1s and 2s, such that

- the order in which the 1s and 2s occur in the sum matters, and
- it is not allowed to have two consecutive 2s.

For example, if  $n = 7$ , then both

$$7 = 1 + 2 + 1 + 2 + 1$$

and

$$7 = 2 + 1 + 1 + 2 + 1$$

are allowed, whereas

$$7 = 1 + 2 + 2 + 1 + 1$$

is not allowed.

Which of the following is true?

- (a)  $S_n = S_{n-1} + S_{n-2}$
  - (b)  $S_n = S_{n-1} + S_{n-3}$
  - (c)  $S_n = S_{n-2} + S_{n-3}$
  - (d)  $S_n = S_{n-1} + S_{n-2} + S_{n-3}$
13. Consider the following recursive algorithm FIB, which takes as input an integer  $n \geq 0$ :

```
Algorithm FIB( $n$ ):  
if  $n = 0$  or  $n = 1$   
  then  $f = n$   
  else  $f = \text{FIB}(n - 1) + \text{FIB}(n - 2)$   
  endif;  
return  $f$ 
```

When running FIB(55), how many calls are there to FIB(50)?

- (a) 6
- (b) 7
- (c) 8
- (d) 9

14. Consider the following recursive algorithm JUSTINBIEBER, which takes as input an integer  $n \geq 1$ , which is a power of 2:

```
Algorithm JUSTINBIEBER( $n$ ):  
if  $n = 1$   
  then order chicken wings  
else if  $n = 2$   
  then drink one pint of India Pale Ale  
  else print “I don’t like Justin Bieber”;  
    JUSTINBIEBER( $n/2$ )  
  endif  
endif
```

For  $n$  a power of 2, let  $B(n)$  be the number of times you print “I don’t like Justin Bieber” when running algorithm JUSTINBIEBER( $n$ ). Which of the following is true?

- (a)  $B(n) = \log n - 1$  for all  $n \geq 2$ .
  - (b)  $B(n) = \log n - 1$  for all  $n \geq 1$ .
  - (c)  $B(n) = \log n$  for all  $n \geq 2$ .
  - (d)  $B(n) = n - 2$  for all  $n \geq 2$ .
15. You flip a fair coin 7 times. Define the event

$A =$  “the result of the first flip is equal to the result of the 7-th flip”.

What is  $\Pr(A)$ ?

- (a)  $\frac{2^5+2}{2^7}$
- (b)  $1/2$
- (c)  $1/3$
- (d)  $1/4$

16. You roll a fair 6-sided die twice. Define the events

$$A = \text{“the sum of the results of the two rolls is 7”}$$

and

$$B = \text{“the result of the first roll is 3”}.$$

Which of the following is true?

- (a)  $\Pr(A) = \Pr(B)$
  - (b)  $\Pr(A) < \Pr(B)$
  - (c)  $\Pr(A) > \Pr(B)$
  - (d) None of the above.
17. Let  $S = \{1, 2, 3, 4, 5, 6, 7\}$ . You choose a uniformly random 3-element subset  $X$  of  $S$ . Thus, each 3-element subset of  $S$  has a probability of  $1/\binom{7}{3}$  of being  $X$ . Define the event

$$A = \text{“4 is an element of } X\text{”}$$

What is  $\Pr(A)$ ?

- (a)  $7/15$
- (b)  $15/7$
- (c)  $3/7$
- (d)  $7/3$





